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The Adjoint of a Differential-Boundary Operator with an Integral Boundary Condition on a Semiaxis*

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1. INTRODUCTION

Differential operators generated by a differential expression

$$ly = -y'' + q(x)\,y$$

and various boundary conditions have been considered extensively in the past few years, for example, by Naimark [8] and Krall [4, 5] on the interval $[0, \infty)$ and by Kemp [2] on the interval $(-\infty, \infty)$. In 1954, Naimark [8] considered the differential operator L_θ generated by the differential expression $ly = -y'' + q(x)\,y$, where $\int_0^\infty e^{\epsilon x} \mid q(x)\mid dx < \infty$ for some $\epsilon > 0$, $0 \leqslant x < \infty$ and the boundary condition $y'(0) - \theta y(0) = 0$, where θ is a fixed complex number. In 1965, Krall [5] considered the differential operator L generated by the differential expression $ly = -y'' + q(x)\,y$, where $\int_0^\infty \mid q(x)\mid dx < \infty$, $0 \leqslant x < \infty$ and the boundary condition $\int_0^\infty K(x)\,y(x)\,dx = \beta y(0) - \alpha y'(0)$, where $K(x)$ is in $L^2(0, \infty)$ and $\mid \alpha \mid^2 + \mid \beta \mid^2 \neq 0$. The adjoint operator L^* of L is generated by

$$lg = -g'' + \overline{q(x)}\,g - \mu_g \overline{K(x)}$$

and the boundary condition $\bar{\beta} g(0) - \bar{\alpha} g'(0) = 0$, where $\mu_g = g(0)/\bar{\alpha}$ or $\mu_g = g'(0)/\bar{\beta}$ or both depending on which is defined. It is not possible for the operator L to be a self-adjoint operator unless $K(x) = 0$ identically on $[0, \infty)$, because the function $K(x)$ appearing in the boundary condition of the operator L arises in the adjoint differential expression and the boundary condition $\bar{\beta} g(0) - \bar{\alpha} g'(0) = 0$ of the adjoint operator L^* has no integral terms. This work of Krall motivates us in this paper to seek a possible self-adjoint operator which generalizes the operator L.

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Consider a differential-boundary expression of the form

$$ly = y'' - h(x)\,(a_1 y(0) + a_2 y'(0)),$$

where $h(x) \in L^2(0, \infty)$, $0 \leqslant x < \infty$ and $|a_1|^2 + |a_2|^2 \neq 0$. Denote by D the set of those functions $f(x)$ defined on $[0, \infty)$ and satisfying

(1) $f(x)$ is in complex $L^2(0, \infty)$,

(2) $f'(x)$ exists and is absolutely continuous on every finite subinterval $[0, b]$ of $[0, \infty)$,

(3) $lf(x)$ is in $L^2(0, \infty)$.

Let $K(x)$ be an arbitrary complex valued function in $L^2(0, \infty)$, and let b_1 and b_2 be complex numbers such that $|b_1|^2 + |b_2|^2 \neq 0$. Denote by D_K the set of those functions $f(x)$ satisfying

(1) $f(x)$ is in D,

(2) $\int_0^\infty K(x) f(x)\, dx = b_1 f(0) - b_2 f'(0)$.

Define the operator L_K by $L_K f(x) = lf(x)$ for all $f(x)$ in D_K.

In this paper we shall discuss the eigenvalues and eigenfunctions of L_K and derive the Green's function of $L_K + \lambda$. Moreover, using the Green's function $G(x, \xi, \lambda)$, we find the adjoint operator $L_K{}^*$ of L_K. The main purpose of this paper is to show how $K(x)$ and $h(x)$ in the operator L_K affect the form of the adjoint operator $L_K{}^*$.

As a remark at the end of this paper, it is explained that L_K is an operator which generalizes the operator L defined by Krall [5].

2. The Solutions of $y'' + \lambda y = h(x)(a_1 y(0, s) + a_2 y'(0, s))$ and Eigenvalues and Eigenfunctions of L_K

LEMMA 1. *If $\zeta = 1 + is\alpha a_2 - \alpha a_1 \neq 0$, where $\lambda^{1/2} = s = \sigma + i\tau$, σ and τ are reals, $\tau > 0$ and*

$$\alpha = \int_0^\infty \frac{e^{isx}}{2is} h(x)\, dx,$$

then $y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s))$ has a solution

$$y(x, s) = e^{isx} + \left[e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi\right] \left[\frac{a_1 + isa_2}{\zeta}\right], \tag{2.1}$$

which is in $L^2(0, \infty)$.

Proof. The homogeneous equation $z'' + \lambda z = 0$ has two linear independent solutions

$$z_1 = e^{isx}, \qquad z_2 = e^{-isx}.$$

Considering $y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s))$ as a nonhomogeneous equation and applying the method of variation of parameters, one obtains

$$\begin{aligned} y(x, s) = c_1 e^{isx} + c_2 e^{-isx} + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi (a_1 y(0, s) + a_2 y'(0, s)) \\ - e^{-isx} \int_0^x \frac{e^{is\xi}}{2is} h(\xi)\, d\xi (a_1 y(0, s) + a_2 y'(0, s)), \end{aligned} \tag{2.2}$$

where c_1 and c_2 are constants. $y(x, s)$ is in $L^2(0, \infty)$ only when

$$c_2 = \int_0^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi (a_1 y(0, s) + a_2 y'(0, s)).$$

Upon substituting c_2 in (2.2), $y(x, s)$ is given by

$$\begin{aligned} y(x, s) = c_1 e^{isx} + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi (a_1 y(0, s) + a_2 y'(0, s)) \\ + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi (a_1 y(0, s) + a_2 y'(0, s)). \end{aligned} \tag{2.3}$$

If $c_1 = 0$, $y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s))$ has either a trivial solution or a non-L^2 solution. Let us set $c_1 \neq 0$. From (2.3), we find

$$y(0, s) = \frac{1 + 2is\alpha a_2}{\zeta} c_1, \qquad y'(0, s) = \frac{is - 2is\alpha a_1}{\zeta} c_1. \tag{2.4}$$

Substituting (2.4) into (2.3) and setting $c_1 = 1$, the equation

$$y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s))$$

has an L^2 solution which is given by

$$y(x, s) = e^{isx} + \left[e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi\right] \left[\frac{a_1 + isa_2}{\zeta}\right]. \tag{2.5}$$

For the proof of $y \in L^2(0, \infty)$, see [8, p. 130].

We note that $y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s))$ has either no solutions or two linear independent solutions,

$$e^{isx} \qquad \text{and} \qquad e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi,$$

when $\zeta = 0$.

We shall discuss in this paper only the case $\zeta \neq 0$ and $a_1 + isa_2 \neq 0$. The case, $\zeta \neq 0$ and $a_1 + isa_2 = 0$, will not be investigated here, since the procedure is similar to the aforementioned case and simple.

THEOREM 1. *For the eigenvalue problem* $L_K y + \lambda y = 0$, *the eigenvalues are* $\lambda = s^2$, *where* $\operatorname{Im} s > 0$ *and* s *is a solution of*

$$2is\delta\zeta + \int_0^\infty K(x)\, v_1(x, s)\, dx(a_1 + isa_2) - (b_1 - isb_2) - 2is\alpha(a_1b_2 + a_2b_1) = 0, \tag{2.6}$$

where

$$\delta = \int_0^\infty \frac{e^{isx}}{2is} K(x)\, dx,$$

and

$$v_1(x, s) = e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi.$$

Proof. Using (2.5) we see that

$$b_1 y(0, s) - b_2 y'(0, s) = \frac{b_1 - isb_2 + 2is\alpha(a_1b_2 + a_2b_1)}{\zeta}. \tag{2.7}$$

Upon substituting (2.5) and (2.7) into the boundary condition,

$$\int_0^\infty K(x)\, y(x)\, dx = b_1 y(0) - b_2 y'(0),$$

one obtains

$$2is\delta\zeta + \int_0^\infty K(x)\, v_1(x, s)\, dx(a_1 + isa_2) - (b_1 - isb_2) - 2is\alpha(a_1b_2 + a_2b_1) = 0, \tag{2.8}$$

and the eigenvalues of L_K are $\lambda = s^2$, where $\operatorname{Im} s > 0$ and s is a solution of (2.8).

The eigenfunctions $y(x, s)$, corresponding to an eigenvalue $\lambda = s^2$, are given by (2.1).

Define a function β of a variable s by

$$\beta = 2is\delta + \frac{a_1 + isa_2}{\zeta} \int_0^\infty K(x)\, v_1(x, s)\, dx - \frac{b_1 - isb_2}{\zeta} - \frac{2is\alpha(a_1b_2 + a_2b_1)}{\zeta}. \tag{2.9}$$

With the function β, (2.8) can be written as

$$\beta\zeta = 0. \tag{2.10}$$

3. The Green's Function for the Operator $L_K + \lambda$

Suppose now that the number $\lambda = s^2$, where $\operatorname{Im} s > 0$, is not a root of the equation $\beta = 0$. Thus, the resolvent R_λ exists. Let the function f lie in the domain of the operator R_λ.

Let

$$R_\lambda f = y.$$

Hence,

$$L_K y + \lambda y = f.$$

Using similar arguments of Lemma 1, the differential-boundary equation $y'' + \lambda y = h(x)\,(a_1 y(0, s) + a_2 y'(0, s)) + f(x)$ has an L^2 solution which is given by

$$\begin{aligned} y(x, s) = A \Big\{ e^{isx} &+ \frac{a_1 + isa_2}{\zeta} \left[e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi \right] \Big\} \\ &+ \frac{a_1 - isa_2}{\zeta} \left[e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi \right] \\ &\times \int_0^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} f(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi, \end{aligned} \tag{3.1}$$

where A is a constant.

From (3.1), we find that

$$\begin{aligned} &b_1 y(0, s) - b_2 y'(0, s) \\ &\quad = \frac{(b_1 - isb_2) + (a_1 b_2 + a_2 b_1)\, 2is\alpha}{\zeta} A + \frac{b_1 + isb_2}{\zeta} \int_0^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi. \end{aligned} \tag{3.2}$$

Upon substituting the expression (3.2) of $b_1 y(0, s) - b_2 y'(0, s)$ and $y(x, s)$ of (3.1) into the boundary condition, $\int_0^\infty K(x)\, y(x)\, dx = b_1 y(0) - b_2 y'(0)$, and solving for A, we have

$$\begin{aligned} A = &- \frac{a_1 - isa_2}{\beta\zeta} \int_0^\infty K(x) \left[e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} h(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} h(\xi)\, d\xi \right] \\ &\times dx \int_0^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi \\ &- \frac{1}{\beta} \int_0^\infty K(x) \left[e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} f(\xi)\, d\xi + e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi \right] dx \\ &+ \frac{b_1 + isb_2}{\beta\zeta} \int_0^\infty \frac{e^{is\xi}}{2is} f(\xi)\, d\xi. \end{aligned} \tag{3.3}$$

Changing the order of integration of the second term of (3.3) and substituting A into (3.1) and simplifying it we have

$$\begin{aligned} y(x, s) = \int_0^\infty \Big[& \frac{\gamma}{2is} e^{isx} e^{is\xi} - \frac{1}{\beta} e^{isx} v_2(\xi, s) + \frac{b_1 + isb_2}{\beta\zeta} \frac{1}{2is} e^{isx} e^{is\xi} \\ & + \frac{\gamma(a_1 + isa_2)}{\zeta} \frac{1}{2is} v_1(x, s) e^{is\xi} - \frac{a_1 + isa_2}{\beta\zeta} v_1(x, s) v_2(\xi, s) \\ & + \frac{1}{2is\beta} \frac{(b_1 + isb_2)(a_1 + isa_2)}{\zeta^2} v_1(x, s) e^{is\xi} + \frac{a_1 - isa_2}{2is\zeta} v_1(x, s) e^{is\xi} \Big] \\ & \times f(\xi)\, d\xi + \int_0^\infty \frac{1}{2is} e^{isx_>} e^{-isx_<} f(\xi)\, d\xi, \end{aligned} \tag{3.4}$$

where

$$v_2(x, s) = e^{-isx} \int_x^\infty \frac{e^{is\xi}}{2is} K(\xi)\, d\xi + e^{isx} \int_0^x \frac{e^{-is\xi}}{2is} K(\xi)\, d\xi,$$

$$\gamma = -\frac{a_1 - isa_2}{\beta\zeta} \int_0^\infty K(x)\, v_1(x, s)\, dx,$$

and

$$e^{isx_>} e^{-isx_<} = \begin{cases} e^{isx} e^{-is\xi}, & x > \xi, \\ e^{is\xi} e^{-isx}, & x < \xi. \end{cases}$$

We shall set

$$\begin{aligned} V_1(x, \xi, \lambda) = & \frac{\gamma}{2is} e^{isx} e^{is\xi} + \frac{\gamma}{2is} \frac{a_1 + isa_2}{\zeta} v_1(x, s) e^{is\xi} \\ & - \frac{1}{\beta} e^{isx} v_2(\xi, s) - \frac{a_1 + isa_2}{\beta\zeta} v_1(x, s) v_2(\xi, s) \\ & + \frac{1}{2is} \frac{b_1 + isb_2}{\beta\zeta} e^{isx} e^{is\xi} + \frac{1}{2is\beta} \frac{(b_1 + isb_2)(a_1 + isa_2)}{\zeta^2} v_1(x, s) e^{is\xi} \\ & + \frac{1}{2is} \frac{a_1 - isa_2}{\zeta} v_1(x, s) e^{is\xi}, \end{aligned} \tag{3.5}$$

and

$$V_2(x, \xi, \lambda) = \frac{1}{2is} e^{isx_>} e^{-isx_<}. \tag{3.6}$$

The Green's function $G(x, \xi, \lambda)$ for $L_K + \lambda$ is given by

$$G(x, \xi, \lambda) = V_1(x, \xi, \lambda) + V_2(x, \xi, \lambda). \tag{3.7}$$

Thus, we have arrived at the following result.

THEOREM 2. *If $L_K y + \lambda y = 0$ has only the trivial solution, then for any function $f(x)$ in $L^2(0, \infty)$, there exists a solution of the equation $L_K y + \lambda y = f$. The solution is expressed by*

$$y(x, s) = \int_0^\infty G(x, \xi, \lambda) f(\xi)\, d\xi.$$

4. THE ADJOINT OPERATOR L_K^* OF L_K

The adjoint operator L_K^* of L_K exists when the domain D_K of L_K is dense in $L^2(0, \infty)$.

THEOREM 3. *If $|b_1|^2 + |b_2|^2 \neq 0$, then the domain D_K of L_K is dense in $L^2(0, \infty)$.*

Proof. See [4, Theorem 1.1].

In addition to the assumptions $\zeta \neq 0$ and $a_1 + isa_2 \neq 0$, we shall assume that $a_1 b_2 + a_2 b_1 \neq 0$. The adjoint operator L_K^* of L_K will be found by using the Green's function $G(x, \xi, \lambda)$.

LEMMA 1. *If g is in the domain of L_K^*, then*

$$\overline{g(\xi)} = \int_0^\infty G(x, \xi, \lambda)\, \overline{(L_K^* + \bar{\lambda})\, g(x)}\, dx. \tag{4.1}$$

Proof. If λ is not an eigenvalue of L_K, we have

$$\begin{aligned}\int_0^\infty &[(L_K + \lambda)\, y(x)]\, \overline{g(x)}\, dx \\ &= \int_0^\infty [(L_K + \lambda)\, y(\xi)] \left[\int_0^\infty G(x, \xi, \lambda)\, \overline{(L_K^* + \bar{\lambda})\, g(x)}\, dx\right] d\xi,\end{aligned} \tag{4.2}$$

for all y in D_K. Since the range of $L_K + \lambda$ is dense in $L^2(0, \infty)$, it follows

$$\overline{g(\xi)} = \int_0^\infty G(x, \xi, \lambda)\, \overline{(L_K^* + \bar{\lambda})\, g(x)}\, dx. \tag{4.3}$$

LEMMA 2. *If g is in the domain of L_K^*, then*

$$\overline{L_K^* g(\xi)} = \overline{g''(\zeta)} + K(\xi) \int_0^\infty \frac{1}{\beta} \left[e^{isx} + \frac{a_1 + isa_2}{\zeta}\, v_1(x, s)\right] \overline{(L_K^* + \bar{\lambda})\, g(x)}\, dx. \tag{4.4}$$

Proof. Differentiating (4.1) twice with respect to ξ, and simplifying it we get

$$\overline{g''(\xi)} = -K(\xi) \int_0^\infty \frac{1}{\beta} \left[e^{isx} + \frac{a_1 + isa_2}{\zeta} v_1(x, s) \right] \overline{(L_K{}^* + \lambda)\, g(x)}\, dx + \overline{L_K{}^* g(\xi)}, \tag{4.5}$$

which yields (4.4) when (4.5) is solved with respect to $\overline{L_K{}^* g(\xi)}$.

LEMMA 3. *If g is in the domain of $L_K{}^*$, then*

$$\begin{aligned} &\int_0^\infty \frac{1}{\beta} \left[e^{isx} + \frac{a_1 + isa_2}{\zeta} v_1(x, s) \right] \overline{(L_K{}^* + \lambda)\, g(x)}\, dx \\ &\qquad = \frac{a_1}{a_1 b_2 + a_2 b_1} \overline{g(0)} + \frac{a_2}{a_1 b_2 + a_2 b_1} \overline{g'(0)}. \end{aligned} \tag{4.6}$$

Proof. Let

$$u_1(x, s) = e^{isx} + \frac{a_1 + isa_2}{\zeta} v_1(x, s). \tag{4.7}$$

Using the definition (4.7), $V_1(x, \xi, \lambda)$ can be written as

$$\begin{aligned} V_1(x, \xi, \lambda) = {} & \frac{\gamma}{2is} u_1(x, s)\, e^{is\xi} - \frac{1}{\beta} u_1(x, s)\, v_2(\xi, s) \\ & + \frac{b_1 + isb_2}{2is\beta\zeta} u_1(x, s)\, e^{is\xi} \\ & + \frac{a_1 - isa_2}{2is(a_1 + isa_2)} u_1(x, s)\, e^{is\xi} - \frac{a_1 - isa_2}{2is(a_1 + isa_2)} e^{isx} e^{is\xi}. \end{aligned} \tag{4.8}$$

Evaluating $\overline{g(0)}$ from (4.1), we find

$$\begin{aligned} \overline{g(0)} = {} & \int_0^\infty u_1(x, s) \frac{1}{\beta} \overline{(L_K{}^* + \lambda)\, g(x)}\, dx \\ & \times \left\{ \frac{\beta\gamma}{2is} - \delta + \frac{b_1 + isb_2}{2is\zeta} + \frac{\beta(a_1 - isa_2)}{2is(a_1 + isa_2)} \right\} \\ & + \int_0^\infty e^{isx} \overline{(L_K{}^* + \lambda)\, g(x)}\, dx \left\{ \frac{a_2}{a_1 + isa_2} \right\}, \end{aligned}$$

and differentiate $\overline{g(\xi)}$ of (4.1) and evaluate $\overline{g'(0)}$, then we have

$$\begin{aligned} \overline{g'(0)} = {} & \int_0^\infty u_1(x, s) \frac{1}{\beta} \overline{(L_K{}^* + \lambda)\, g(x)}\, dx \\ & \times \left\{ \frac{\beta\gamma}{2} + is\delta + \frac{b_1 + isb_2}{2\zeta} + \frac{\beta(a_1 - isa_2)}{2(a_1 + isa_2)} \right\} \\ & + \int_0^\infty e^{isx} \overline{(L_K{}^* + \lambda)\, g(x)}\, dx \left\{ \frac{-a_1}{a_1 + isa_2} \right\}. \end{aligned} \tag{4.9}$$

Let

$$D_{11} = \frac{\beta\gamma}{2is} - \delta + \frac{b_1 + isb_2}{2is\zeta} + \frac{\beta(a_1 - isa_2)}{2is(a_1 + isa_2)},$$

$$D_{21} = \frac{\beta\gamma}{2} + is\delta + \frac{b_1 + isb_2}{2\zeta} + \frac{\beta(a_1 - isa_2)}{2(a_1 + isa_2)},$$

$$D_{12} = \frac{a_2}{a_1 + isa_2},$$

$$D_{22} = \frac{-a_1}{a_1 + isa_2}.$$

Consequently, (4.9) can be written as

$$\overline{g(0)} = D_{11} \int_0^\infty \frac{1}{\beta} u_1(x, s) \overline{(L_K^* + \bar{\lambda}) g(x)}\, dx + D_{12} \int_0^\infty e^{isx} \overline{(L_K^* + \bar{\lambda}) g(x)}\, dx,$$

and

$$\overline{g'(0)} = D_{21} \int_0^\infty \frac{1}{\beta} u_1(x, s) \overline{(L_K^* + \bar{\lambda}) g(x)}\, dx + D_{22} \int_0^\infty e^{isx} \overline{(L_K^* + \bar{\lambda}) g(x)}\, dx. \tag{4.10}$$

Calculating the coefficient determinant

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}$$

of the simultaneous equation (4.10), it yields

$$D_{11}D_{22} - D_{21}D_{12} = -\frac{a_1b_2 + a_2b_1}{a_1 + isa_2},$$

and solving (4.10) for the first integral, we get

$$\int_0^\infty \frac{1}{\beta} u_1(x, s) \overline{(L_K^* + \bar{\lambda}) g(x)}\, dx = \frac{a_1}{a_1b_2 + a_2b_1} \overline{g(0)} + \frac{a_2}{a_1b_2 + a_2b_1} \overline{g'(0)},$$

which completes the lemma.

Combining the result of Lemma 3 with Lemma 2, we have the following.

LEMMA 4. *If g is in the domain of L_K^*, we have*

$$\overline{L_K^* g(\xi)} = \overline{g''(\xi)} + K(\xi) \left\{ \frac{a_1}{a_1b_2 + a_2b_1} \overline{g(0)} + \frac{a_2}{a_1b_2 + a_2b_1} \overline{g'(0)} \right\}. \tag{4.11}$$

We are now in a position to find the condition for which the function g in the domain of the operator $L_K{}^*$ satisfies. Multiplying the first equation of (4.10) by

$$-\frac{b_1}{a_1b_2+a_2b_1},$$

the second equation by

$$\frac{b_2}{a_1b_2+a_2b_1},$$

and adding, we obtain

$$-\frac{b_1}{a_1b_2+a_2b_1}\overline{g(0)}+\frac{b_2}{a_1b_2+a_2b_1}\overline{g'(0)}$$
$$=\int_0^\infty\left[\frac{2is\delta-b_1+isb_2}{\beta(a_1+isa_2)}u_1(x,s)-\frac{1}{a_1+isa_2}e^{isx}\right]\overline{(L_K{}^*+\bar{\lambda})g(x)}\,dx. \tag{4.12}$$

Let

$$F(x,s)=\frac{2is\delta-b_1+isb_2}{\beta(a_1+isa_2)}u_1(x,s)-\frac{1}{a_1+isa_2}e^{isx}. \tag{4.13}$$

It is easily shown that $F(x, s)$ is in D. If $F(x, s)$ is in D_K, the right side of (4.12) can be written as

$$\int_0^\infty[(L_K+\lambda)F(x,s)]\,\overline{g(x)}\,dx,$$

for all g in $D_K{}^*$. We wish to show that $F(x, s)$ is in D_K.

LEMMA 5. *$F(x, s)$ is in D_K.*

Proof. Using (4.13), one obtains

$$b_1F(0,s)-b_2F'(0,s)$$
$$=\frac{(-b_1+isb_2+2is\delta)\,[b_1-isb_2+(a_1b_2+a_2b_1)\,2is\alpha]}{\beta(a_1+isa_2)\,\zeta}-\frac{b_1-isb_2}{a_1+isa_2}, \tag{4.14}$$

and

$$\int_0^\infty K(x)\,F(x,s)\,dx$$
$$=\frac{(-b_1+isb_2+2is\delta)\,[b_1-isb_2+(a_1b_2+a_2b_1)\,2is\alpha]}{\beta(a_1+isa_2)\,\zeta}-\frac{b_1-isb_2}{a_1+isa_2}, \tag{4.15}$$

which shows that $F(x, s)$ satisfies the boundary condition

$$\int_0^\infty K(x)\,y(x)\,dx = b_1 y(0) - b_2 y'(0);$$

thus, $F(x, s)$ is in D_K.

Applying $F(x, s)$ to the operator $L_K + \lambda$, it yields $h(x)$, i.e.,

$$(L_K + \lambda)\,F(x, s) = h(x). \tag{4.16}$$

From Lemma 5 and (4.16), we conclude that

$$\begin{aligned}\int_0^\infty F(x, s)\,\overline{(L_K{}^* + \bar{\lambda})\,g(x)}\,dx &= \int_0^\infty [(L_K + \lambda)\,F(x, s)]\,\overline{g(x)}\,dx \\ &= \int_0^\infty h(x)\,\overline{g(x)}\,dx.\end{aligned} \tag{4.17}$$

Combining (4.17) with (4.12), we have the following lemma.

LEMMA 6. *If g is in the domain of $L_K{}^*$, then g satisfies*

$$\int_0^\infty h(x)\,\overline{g(x)}\,dx = -\frac{b_1}{a_1b_2 + a_2b_1}\,\overline{g(0)} + \frac{b_2}{a_1b_2 + a_2b_1}\,\overline{g'(0)}. \tag{4.18}$$

With the lemmas proved above the following results are obtained.

THEOREM 4. *The adjoint operator $L_K{}^*$ of L_K is generated by the expression*

$$l^*g = g''(x) + \overline{K(x)}\left\{\frac{\bar{a}_1}{\overline{a_1b_2 + a_2b_1}}\,g(0) + \frac{\bar{a}_2}{\overline{a_1b_2 + a_2b_1}}\,g'(0)\right\}$$

and the boundary condition

$$\int_0^\infty \overline{h(x)}\,g(x)\,dx = -\frac{\bar{b}_1}{\overline{a_1b_2 + a_2b_1}}\,g(0) + \frac{\bar{b}_2}{\overline{a_1b_2 + a_2b_1}}\,g'(0). \tag{4.19}$$

COROLLARY 1. *The operator L_K is self-adjoint if and only if*

$$a_1b_2 + a_2b_1 = -1;$$

a_1, a_2, a_3 and a_4 are reals; and $h(x) = \overline{K(x)}$.

The Green's function $G(x, \xi, \lambda)$ derived in this paper satisfies the homogeneous equation $y'' + \lambda y = h(x)\,(a_1 y(0) + a_2 y'(0))$ and the boundary condition

$$\int_0^\infty K(x)\,y(x)\,dx = b_1 y(0) - b_2 y'(0),$$

as a function of x.

EXAMPLE. On the interval $[0, \infty)$, consider the system

$$ly = y'' - e^{-x}(2y(0) + y'(0)), \qquad \int_0^\infty e^{-x}y(x)\,dx = y(0) + y'(0). \tag{4.20}$$

It is a self-adjoint system. Let us define the operator $L_{e^{-x}}$ by $L_{e^{-x}}y = ly$ for all y in $L^2(0, \infty)$, which satisfy the boundary condition of (4.20).

The Green's function $G(x, \xi, \lambda) = G(x, \xi, -1)$ for the operator $L_{e^{-x}} - 1$, where $\lambda = -1$ is not an eigenvalue of $L_{e^{-x}}$, is given by

$$G(x, \xi, \lambda) = G(x, \xi, -1) = V_1(x, \xi, -1) + V_2(x, \xi, -1),$$

where

$$V_1(x, \xi, -1) = -\tfrac{1}{2}\,e^{-x}e^{-\xi} + \tfrac{2}{3}\,xe^{-x}e^{-\xi} + \tfrac{2}{3}\,e^{-x}\xi e^{-\xi} - \tfrac{2}{9}\,xe^{-x}\xi e^{-\xi},$$

and

$$V_2(x, \xi, -1) = -\tfrac{1}{2}\,e^{-x_>}e^{x_<}.$$

Remark. The results obtained in this paper can be extended over the case $ly = y'' - q(x)\,y - h(x)\,(a_1y(0) + a_2y'(0))$, where $\int_0^\infty |\,q(x)|\,dx < \infty$ and $h(x)$ is in $L^2(0, \infty)$, merely by substituting appropriate functions for e^{isx} and e^{-isx} throughout.

Consider the two equations $y'' - q(x)\,y + \lambda y = 0$ and $y'' + \lambda y = 0$, where $\lambda = s^2$, $s = \sigma + i\tau$, σ and τ are reals and $\tau > 0$. $y'' - q(x)\,y + \lambda y = 0$, has two linear independent solutions $y_1(x, s)$ and $y_2(x, s)$. As $x \to \infty$, we have $y_1(x, s) = e^{isx}[1 + o(1)]$, $y_1'(x, s) = e^{isx}[is + o(1)]$, $y_2(x, s) = e^{-isx}[1 + o(1)]$ and $y_2'(x, s) = e^{-isx}[-is + o(1)]$. The solutions $y_1(x, s)$ and $y_2(x, s)$ have the same behavior as e^{isx} and e^{-isx} respectively, as $x \to \infty$. The Wronskians $W(y_1(x, s), y_2(x, s))$ and $W(e^{isx}, e^{-isx})$ are both equal to $-2is$. Therefore any of the results for the case $ly = y'' - h(x)\,(a_1y(0) + a_2y'(0))$ can be translated to analogous results for the case

$$ly = y'' - q(x)\,y - h(x)\,(a_1y(0) + a_2y'(0))$$

simply by substituting $y_1(x, s)$ for each occurrence of e^{isx} and $y_2(x, s)$ for each occurrence of e^{-isx}.

For example, substituting $y_1(x, s)$ for e^{isx} and $y_2(x, s)$ for e^{-isx} in (2.1), we obtain an L^2 solution

$$y(x, s) = y_1(x, s) + \left[y_2(x, s)\int_x^\infty \frac{y_1(\xi, s)}{2is}\,h(\xi)\,d\xi + y_1(x, s)\int_0^x \frac{y_2(\xi, s)}{2is}\,h(\xi)\,d\xi\right] \times \left[\frac{a_1 + isa_2}{\zeta}\right],$$

where

$$\zeta = 1 + isa_2 \int_0^\infty \frac{y_1(x, s)}{2is} h(x)\, dx - a_1 \int_0^\infty \frac{y_1(x, s)}{2is} h(x)\, dx,$$

of the equation $y'' - q(x)\, y + \lambda y = h(x)\, (a_1 y(0, s) + a_2 y'(0, s))$.

Consequently, this paper can be considered to be an extension of the work of Krall and for simplicity, $q(x) = 0$ was chosen in this paper.

References

1. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill Co., New York, 1955.
2. A. M. Kemp, A singular boundary value problem for a non-self adjoint differential operator, *Canad. J. Math.* **10** (1958), 447–462.
3. Taeboo Kim, Investigation of a differential-boundary operator of the second order with an integral boundary condition on a semi-axis, doctoral dissertation, The Pennsylvania State University, University Park, PA 16082, 1969.
4. A. M. Krall, A nonhomogeneous eigenfunction expansion, *Trans. Amer. Math. Soc.* **117** (1965), 352–361.
5. A. M. Krall, The adjoint of a differential operator with integral boundary conditions, *Proc. Amer. Math. Soc.* **16** (1965), 738–742.
6. A. M. Krall, Differential operators and their adjoint under integral and multiple point boundary conditions, *J. Differential Equations* **4** (1968), 327–336.
7. A. M. Krall, Nonhomogeneous differential operators, *Michigan Math. J.* **12** (1965), 247–255.
8. M. A. Naimark, Investigation of the spectrum and the expansion in eigenfunctions of a non-self adjoint differential operator of the second order on a semi-axis, *Trudy Moskov. Math. Obšč.* **3** (1954), 181–270; English translation, *Amer. Math. Soc. Transl.* **16** (1960), 103–194.
9. M. A. Naimark, "Linear Differential Operators," Part I, Frederic Ungar, New York, 1967.
10. I. Stakgold, "Boundary Value Problems of Mathematical Physics," Vol. I, Macmillan Co., New York, 1967.
11. W. M. Whyburn, Differential systems with general boundary conditions, *Univ. of Cal. Pub. in Math.* **2** (1944), 45–61.